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Decomposition Phenomenon in One-Dimensional Scope-Three Tessellation Automata with Arbitrary Number of States

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A decomposition phenomenon peculiar to the tessellation automata (TA) is investigated. The result is that, in scope-three TA with an arbitrary number of states, any finite pattern can be transformed to a standard pattern by repeated applications of inverse maps of two specific global maps.

1. INTRODUCTION

The tessellation automaton (TA) is a formalization of the concept of an infinite regular array of identical finite-state machines uniformly interconnected in the sense that each machine can directly receive information by means of interconnecting wires from a finite number of neighboring machines where the spatial arrangement of these neighboring machines is the same relative to each machine in the array. Each machine can synchronously change its state at discrete time steps as a function of the states of the neighboring machines. This function can change from time step to time step, but will be identical for each machine in the array at any given time step. The simultaneous action of these local functions will define global functions which will act on the entire array changing patterns of machine states in the array to other patterns.

Such arrays have been applied in such diverse areas as pattern recognition, e.g., Unger (1958, 1959), Beyer (1970), and Smith (1971); machine self-reproduction, e.g., von Neumann (1966) and Codd (1968) and Arbib (1966); and evolution theory, e.g., Barricelli (1957, 1963).

Among these works, Yamada and Amoroso have studied a fundamental problem concerning behavior of such automata, namely: For an arbitrarily given tessellation automaton, can all the (finite) state patterns of the array be reached by some sequence of global transformations from a certain canonical starting pattern? We shall refer to this as the completeness problem for tessellation automata. The completeness problem has been solved in the case for one-dimentional TA with contiguous neighborhood structure (Yamada and Amoroso (1970), Maruoka and Kimura, Kubo and Kimura (1972), and Nasu and Honda (1976)).

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The work on this problem has led to a pattern decomposition result which states that there are TA that operate repeatedly by applying one transformation which will decompose any pattern placed in the array into atomic pieces in a uniform way. As Yamada and Amoroso (1970) mentioned, the pattern decomposition phenomenon peculiar to TA have a good deal of intrinsic interest. In this paper, we derive a result about the decomposition phenomenon observed in three-scope TA with any number of states. This result is a generalization of the results obtained by Yamada and Amoroso (1970) and Kubo and Kimura (1972). Furthermore, from the results we derive a result about the completeness problem for one-dimensional TA, which has been obtained by Nasu and Honda independently employing graph techniques. The proof techniques of these results are similar to those of Yamada and Amoroso (1970) and the authors' previous one (1974), hence it helps to read their papers before the present one. It seems to be interesting that we can also apply the decomposition result to a proof of the completeness problem for multidimentional TA (Maruoka and Kimura).

2. The Decomposition Phenomenon and the Completeness Problem for TA

In the main we employ the definition and notation given by Yamada and Amoroso (1970).

The abbreviation TA is used to mean tessellation automaton. By a one-dimensional, q-state, scope-n TA, we mean TA of the form

$$(\Sigma, Z, X, T_F),$$

where $\Sigma = \{0, 1, 2,..., q-1\}$ is the set of states that can be assumed by any cell (finite-state machine) in the one-dimensional array. Z is the set of integers which is used to name the cells of the one-dimentional array. $X = ((-1), (0), (1), (2),..., (n-2))$ is called the neighborhood index of the TA and is used to define the uniform interconnection pattern. A configuration is defined as a mapping c: $Z \to \Sigma$. C denotes the set of all configurations with respect to Z and Σ. The image of i under $c \in C$ is written $c(i)$ and will be referred to as "the contents of cell i in configuration c." Given scope-n TA, a mapping σ: $\Sigma^n \to \Sigma$ is called a local map. A global map τ: $C \to C$ is defined from local map σ as

$$c\tau = c' \qquad \text{if and only if for any } i \in Z,$$
$$c'(i) = \sigma(c(i-1), c(fi), c(i+1),..., c(i+n-2)).$$

Such global maps defined in this way will also be called parallel maps on the set of configurations. Usually $0 \in \Sigma$ denotes the quiescent state. A configuration c

is called finite if and only if $c(i) = 0$ for all but finitely many cell i. The set of all finite configurations for a given TA is denoted by C_F. It is easy to see that C_F is closed for a given global map τ (i.e., for any $c \in C_F$, $c\tau$ is also in C_F) if and only if $\sigma(0,..., 0) = 0$ where σ is the local map defining τ. T_F denotes the set of all the finite-configuration preserving parallel maps for the TA. If necessary, we write $T_F^{(q,n)}$ to show that we are considering the case with q-state and scope-n. Let us denote the one-dimensional q-state scope-n TA by $\mathfrak{A}^{(q,n)}$.

Let $S \subseteq T_F^{(q,n)}$. For $c, c' \in C_F$ we write

$$c \vdash c' \text{ in } S,$$

if $c = c'$ or there exist a sequence of configurations $c_0, c_1, ..., c_m$, $m > 0$, and a sequence of global functions $\tau_0, ..., \tau_{m-1}$ in S such that $c_i\tau_i = c_{i+1}$ for all i, $0 \leqslant i < m$, and $c_0 = c$, $c_m = c'$. Let us define c_1 and c_2 as shift-equivalent if and only if there exists a $k \in Z$, such that for any $i \in Z$, $c_1(i + k) = c_2(i)$. The equivalence classes on C_F determined by the relation of shift-equivalence are called finite patterns, or just patterns. $[c]$ denotes the pattern containing configuration c. Let c_p be the configuration such that $c_p(0) = 1$ and $c_p(i) = 0$ for all $i \in Z$ with $i \neq 0$. We adopt c_p as the starting configuration. Let

$$\mathfrak{A}^{(q,n)}(c_p) = \{[c] \mid c_p \vdash c \text{ in } T_F^{(q,n)}\}.$$

The completeness problem for $\mathfrak{A}^{(q,n)}$ is then the question whether or not $\mathfrak{A}^{(q,n)}(c_p) = \mathfrak{P}^{(q)}$, where $\mathfrak{P}^{(q)}$ is the set of all finite patterns for one-dimensional TA with the set of states $\Sigma = \{0, 1,..., q - 1\}$. Note that $\mathfrak{A}^{(q,n)} = \mathfrak{P}^{(q)}$ is equivalent to $c \vdash c'$ in $T_F^{(q,n)}$ for any $c, c' \in C_F$ with $c \neq \bar{0}$, where $\bar{0}$ denotes the configuration with no nonquiescent state. If follows from the fact that for any $c \neq \bar{0}$, $c \vdash c_p$ in $T_F^{(q,n)}$.

Yamada and Amoroso (1970) established the decomposition theorem, which is concerned with $\mathfrak{A}^{(2,3)}$. By a configuration in standard decomposition they mean one whose pattern is of the form

$$\cdots 0001100^*1100^* \cdots 1100^*1000^*\ 1000^* \cdots 1000^*1000 \cdots,$$

where each occurrence of 0^* is any finite sequence of zeros, including the sequence of zero length. The decomposition theorem for the case of $\mathfrak{A}^{(2,3)}$ is stated as follows. That is, there exists τ_1 and $\tau_2 \in T_F^{(2,3)}$ such that for any $c \in C_F$ that exist c' in standard decomposition with $c' \vdash c$ in $\{\tau_1, \tau_2\}$. From this theorem it follows that in order to verify $\mathfrak{A}^{(2,4)}(c_p) = \mathfrak{P}^{(2)}$, it is sufficient to show $c_p \vdash c'$ in $T_F^{(2,4)}$ for any c' in standard decomposition. In this way Yamada and Amoroso established $\mathfrak{A}^{(2,4)}(c_p) = \mathfrak{P}^{(2)}$. It was along this line that authers proved $\mathfrak{A}^{(2,3)}(c_p) = \mathfrak{P}^{(2)}$. Kubo and Kimura generalized the decomposition theorem to the case where $q = 3$, $n = 3$ and succeed to prove $\mathfrak{A}^{(3,3)}(c_p) = \mathfrak{P}^{(3)}$.

In this paper, the decomposition theorem is further generalized to the case where $n = 3$ and q is arbitrary. In view of the generalized decomposition theorem, we shall also establish $\mathfrak{A}^{(q,3)}(c_p) = \mathfrak{P}^{(q)}$ for any $q \geqslant 2$.

3. The Generalized Decomposition Theorem

Define $\sigma_1 : \Sigma^3 \to \Sigma$ as

$$\begin{aligned}
\sigma_1(\alpha\beta\gamma) &= 0, && \text{if } \beta = \gamma \text{ and } \alpha \neq \beta;\\
&= \beta, && \text{if } \alpha = \beta = \gamma;\\
&= \gamma, && \text{if } \beta \neq \gamma \text{ and } \gamma \neq 0;\\
&= 0, && \text{if } \beta \neq \gamma, \gamma = 0, \text{ and } \alpha = \beta;\\
&= \beta, && \text{if } \beta \neq \gamma, \gamma = 0, \text{ and } \alpha \neq \beta.
\end{aligned}$$

$\sigma_2 : \Sigma^3 \to \Sigma$ is defined as

$$\sigma_2(\alpha\beta\gamma) = \nu \Leftrightarrow \sigma_1(\gamma\beta\alpha) = \nu,$$

Note that σ_1 and σ_2 defined above are identical with σ_{166} and σ_{180}, respectively, defined by Yamada and Amoroso (1970) when $q = 2$, and that they are identical with σ_2 and σ_1, respectively, defined by Kubo and Kimura (1972) when $q = 3$. Let τ_1 and τ_2 be the global maps defined from σ_1 and σ_2, respectively.

Put

$$\begin{aligned}
L^{(q)} = \{&a_1{}^2a_2{}^2 \cdots a_i{}^2 0^m \mid a_1, \ldots, a_i \in \{1, \ldots, q-1\}, i \geqslant 1,\\
&m \geqslant 1, a_j \neq a_{j+1} \text{ for any } j, 1 \leqslant j \leqslant i-1\},
\end{aligned}$$

$$\begin{aligned}
R^{(q)} = \{&0^m 0a_i 0a_{i-1} \cdots 0a_1 \mid a_1, \ldots, a_i \in \{1, \ldots, q-1\}, i \geqslant 1,\\
&m \geqslant 1, a_j \neq a_{j+1} \text{ for any } j, 1 \leqslant j \leqslant i-1\}.
\end{aligned}$$

In this section, we derive the next theorem, which will be referred to as the decomposition theorem.

Theorem[1] 1. *For any $c_0 \in C_F$, there exist $\xi \in \{\tau_1, \tau_2\}^*$ and $c \in C_F$ such that*

$$c = c_0\xi^{-1}$$

and $[c] = \bar{0}\eta_1 \cdots \eta_r\zeta_s \cdots \zeta_1\bar{0}$, *where* $\eta_1, \ldots, \eta_r \in L^{(q)}$, $\zeta_1, \ldots, \zeta_s \in R^{(q)}$, $r \geqslant 0$. $s \geqslant 0$.

[1] Yamada and Amoroso (1970) proved Theorem 1 for the case with $q = 2$ and Kubo and Kimura (1972) proved the theorem for the case with $q = 3$.

We represent patterns as follows. If $c \in C_{\mathrm{F}}$ is such that $c(k+i) = \alpha_i$, $1 \leqslant i \leqslant j$, for some integer k and positive integer j, and if $c(k+i) = 0$ for all $i < 1$ and $i > j$, then $[c]$ can be represented by

$$\bar{0}\alpha_1\alpha_2 \cdots \alpha_j\bar{0}.$$

If we are unconcerned with the symbols between α_l and α_r then we would write the pattern as

$$\bar{0}\alpha_1\alpha_2 \cdots \alpha_l \text{ ——— } \alpha_r\alpha_{r+1} \cdots \alpha_j\bar{0}.$$

We shall use these conventions in what follows. Before we proceed to prove the decomposition theorem, we give an example of the decomposition phenomenon that helps to understand the following results.

Example. Let $q = 3$. Given c_0 as following, Theorem 1 holds if we put $\xi = (\tau_2\tau_1)^i$ for any $i \geqslant 6$.

$$[c_{12}] = [c_{11}\tau_2^{-1}], \quad \bar{0}\ \underline{2\ 2\ 0}\ \underline{1\ 1\ 0}\ 0\ 0\ 0\ \underline{0\ 0\ 1}\ \underline{0\ 0\ 1\ 0\ 2\ 0\ 1}\ \bar{0}$$

$$[c_{11}] = [c_{10}\tau_1^{-1}], \quad \bar{0}\ 2\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 2\ 2\ 1\ 1\ \bar{0}$$

$$[c_{10}] = [c_9\tau_2^{-1}], \quad \bar{0}\ \underline{2\ 2\ 0}\ \underline{1\ 1\ 0}\ 0\ \underline{0\ 0\ 1}\ \underline{0\ 0\ 1\ 0\ 2\ 0\ 1}\ \bar{0}$$

$$[c_9] = [c_8\tau_1^{-1}], \quad \bar{0}\ 2\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 2\ 2\ 1\ 1\ \bar{0}$$

$$[c_8] = [c_7\tau_2^{-1}], \quad \bar{0}\ \underline{2\ 2\ 0}\ \underline{1\ 1\ 0\ 0\ 1}\ \underline{0\ 0\ 1\ 0\ 2\ 0\ 1}\ \bar{0}$$

$$[c_7] = [c_6\tau_1^{-1}], \quad \bar{0}\ 2\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 2\ 2\ 1\ 1\ \bar{0}$$

$$[c_6] = [c_5\tau_2^{-1}], \quad \bar{0}\ \underline{2\ 2\ 0}\ 1\ 1\ 1\ \underline{0\ 0\ 1\ 0\ 2\ 0\ 1\ \bar{0}}$$

$$[c_5] = [c_4\tau_1^{-1}], \quad \bar{0}\ 2\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 2\ 2\ 1\ 1\ \bar{0}$$

$$[c_4] = [c_3\tau_2^{-1}], \quad \bar{0}\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 0\ 2\ 0\ 1\ \bar{0}$$

$$[c_3] = [c_2\tau_1^{-1}], \quad \bar{0}\ 2\ 1\ 1\ 1\ 0\ 1\ 2\ 2\ 1\ 1\ \bar{0}$$

$$[c_2] = [c_1\tau_2^{-1}], \quad \bar{0}\ 2\ 1\ 0\ 1\ 0\ 1\ 2\ 0\ 1\ \bar{0}$$

$$[c_1] = [c_0\tau_1^{-1}], \quad \bar{0}\ 2\ 2\ 1\ 1\ 1\ 1\ 1\ 2\ 1\ 1\ \bar{0}$$

$$[c_0], \quad \bar{0}\ 2\ 0\ 1\ 0\ 1\ 1\ 1\ 2\ 1\ \bar{0}$$

In the example, we underline the occurrence of the sequences in $L^{(3)}$ or $R^{(3)}$.

The next theorem says that global maps τ_1 and τ_2 restricted to finite configurations are bijections.

THEOREM 2. *For any $c \in C_F$ there exists a unique $c_1 \in C_F$ such that $c_1\tau_1 = c$ and a unique $c_2 \in C_F$ such that $c_2\tau_2 = c$.*

Proof. (i) Let $c \in C_F$ and let l be the least integer such that $c(l) \neq 0$. That is, the leftmost nonquiescent state of c is in cell l. From now on, we abbreviate the cell l or the content of cell l by a_L. Since $\sigma_1(\alpha\beta\gamma) = \gamma$ for $\beta \neq \gamma$ and $\gamma \neq 0$, we have $c_1(j) = 0$ for any $j \leqslant l$.

(ii) From the definition of τ_1, it can be verified that, given $\alpha, \beta, \nu \in \Sigma$, there exists a unique γ such that $\sigma_1(\alpha\beta\gamma) = \nu$. Such γ is given as

$$\begin{aligned} \gamma &= \nu, && \text{if } \alpha = \beta; \\ &= 0, && \text{if } \alpha \neq \beta \text{ and } \nu = \beta; \\ &= \beta, && \text{if } \alpha \neq \beta,\ \nu \neq \beta \text{ and } \nu = 0; \\ &= \nu, && \text{if } \alpha \neq \beta,\ \nu \neq \beta \text{ and } \nu \neq 0. \end{aligned}$$

(iii) Let r be the largest integer such that $c(r) \neq 0$. That is, the rightmost nonquiescent state of c is in cell r. From now on, we abbreviate the cell r or the content of cell r by a_R. From the definition of τ_1, we can verify that $c_1(j) = 0$ for any $j > r + 2$.

From (i), (ii), and (iii), the theorem is established for τ_1. A similar argument proves the case for τ_2. Q.E.D.

Now we introduce the notion of framing which will be useful to determine c_1 (c_2) from c such that $c_1\tau_1 = c$ ($c_2\tau_1 = c$). To understand Definitions 1, 2, 3, and 4, it will be helpful to refer to the example after Definition 4.

DEFINITION 1. Let A be a nonempty subset of Σ^*. A sequence of pairs of integers $((t_1, e_1), (t_2, e_2), \ldots, (t_k, e_k))$, $k \geqslant 0$, is called a *left framing* of $c \in C_F$ with respect to A if and only if the following four conditions are satisfied.[2]

(1) If a_L in c is in cell l, $t_1 = l$ holds.

(2) $t_i \leqslant e_i$ for $i = 1, \ldots, k$.

(3) $e_i + 1 = t_{i+1}$ for $i = 1, \ldots, k - 1$.

(4) $c(t_i)\, c(t_i + 1) \cdots c(e_i) \in A$ for $i = 1, \ldots, k$.

DEFINITION 2. Let A be a nonempty subset of Σ^*. A sequence of pairs of integers $((e_k, t_k), \ldots, (e_2, t_2), (e_1, t_1))$, $k \geqslant 0$, is called a *right framing* of $c \in C_F$ with respect to A if and only if the following four conditions are satisfied.

[2] Note that the null sequence of pairs of integers is regarded as a left framing. This is also the case with the right framing defined later.

(1) If a_R in c is in cell r, $t_1 = r$ holds.

(2) $t_i \geqslant c_i$ for $i = 1,\ldots, k$.

(3) $e_i + 1 = t_{i+1}$ for $i = 1,\ldots, k - 1$.

(4) $c(t_i)\, c(t_i + 1) \cdots c(e_i) \in A$ for $i = 1,\ldots, k$.

Each pair of integers that composes the left framing or right framing is called a *left frame* or *right frame*, respectively. When (t_i, e_i) is a left frame, t_i is called the *head* of left frame (t_i, e_i). Similarly when (e_i, t_i) is a right frame, t_i is called the *head* of right frame (e_i, t_i).

For $a \in \{1, 2,\ldots, q - 1\}$, let

$$N_1{}^a = \{a0a^i0^j \mid i \geqslant 1, j \geqslant 0\},$$

$$N_2{}^a = \{a00^j \mid j \geqslant 0\},$$

$$N_3{}^a = \{aa0^j \mid j \geqslant 0\},$$

$$N_4{}^a = \{a\},$$

$$N = \bigcup_{l=1}^{4} \bigcup_{a=1}^{q-1} N_l{}^a,$$

where $\cup$ denotes the union of sets.[3]

DEFINITION[4] 3. Let a_R in $c \in C_F$ be in cell r. $((t_1, e_1),\ldots, (t_k, e_k))$ is a τ_1-*framing* of c if and only if the following three conditions are satisfied.

(1) $((t_1, e_1),\ldots, (t_k, e_k))$ is a left framing of c with respect to N.

(2) $e_k = r + 1$.

(3) For any i, $1 \leqslant i \leqslant k - 1$ and any j, $j > e_i$, $c(t_i)\, c(t_i + 1) \cdots c(j) \in N$.

Let $\xi = \alpha_1\alpha_2 \cdots \alpha_i \in \Sigma^*$, where $\alpha_1, \ldots, \alpha_i \in \Sigma$. Reverse of ξ, denoted by ξ^R, is defined to be $\alpha_i\alpha_{i-1} \cdots \alpha_1$. If $S \subset \Sigma^*$, then $S^R = \{\xi^R \mid \xi \in S\}$.

DEFINITION[4] 4. Let $c \in C_F$ and let a_L in c be in cell l. $((e_k, t_k),\ldots, (e_1, t_1))$ is a τ_2-framing of c if and only if the following three conditions are satisfied.

(1) $((e_k, t_k),\ldots, (e_1, t_1))$ is a right framing of c with respect to N^R.

(2) $e_k = l - 1$.

(3) For any i, $1 \leqslant i \leqslant k - 1$ and any j, $j < e_i$, $c(j)\, c(j + 1) \cdots c(t_i) \in N^R$.

[3] ξ^i denotes the sequence

$$\overbrace{\xi\xi \cdots \xi}^{i\text{-times}}.$$

In particular, ξ^0 denotes the null sequence λ, where $\xi \in \Sigma^*$.

[4] The reason why we use τ_1 and τ_2 in these definitions will be evident, consult Lemmas 5 and 6.

A frame of a τ_1-framing or a τ_2-framing is called a τ_1-frame or a τ_2-frame, respectively.

EXAMPLE. Let c be the configuration such that

$$[c] = \bar{0}\ 2\ 1\ 2\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 2\ 2\ \bar{0}$$

and a_L in c is in cell 1. Then it is easily seen that a τ_1-framing of c is

$$((1, 1), (2, 2), (3, 3), (4, 5), (6, 13), (14, 16), (17, 19))$$

and that a τ_2-framing of c is

$$(0, 1), (2, 2), (3, 3), (4, 4), (5, 6), (7, 14), (15, 18)).$$

From the definition of τ_1 we can see that relative positions of $[c]$ and $[c\tau_1^{-1}]$ are

$$\begin{aligned}
[c\tau_1^{-1}] &= \bar{0}\ 0\ 0\ 2\ 1\ 2\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 2\ \bar{0},\\
[c] &= \bar{0}\ 0\ \dot{2}\ \dot{1}\ \dot{2}\ \dot{1}\ 1\ \dot{1}\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ \dot{1}\ 0\ 0\ \dot{2}\ 2\ \bar{0},
\end{aligned}$$

where $\cdot$ is placed above the heads of τ_1-frames. Note that a symbol in $[c\tau_1^{-1}]$ is written directly above a symbol in $[c]$ if these symbols represent the contents of the same cell for $c\tau_1^{-1}$ and c. Similarly, relative positions of $[c]$ and $[c\tau_1^{-1}]$ are

$$\begin{aligned}
[c\tau_2^{-1}] &= \bar{0}\ 2\ 2\ 1\ 2\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 2\ 0\ \bar{0},\\
[c] &= \bar{0}\ 0\ 0\ \dot{2}\ \dot{1}\ \dot{2}\ \dot{1}\ 1\ \dot{1}\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ \dot{1}\ 0\ 0\ 2\ \dot{2}\ \bar{0},
\end{aligned}$$

where $\cdot$ is placed above the heads of τ_2-frames.

The next lemma can be easily verified.

LEMMA 3. *For any $c \in C_F$, there exist a unique τ_1-framing and a unique τ_2-framing of c.*

From the example above, we may expect the following two lemmas to hold.

LEMMA 4. *Let $c_1 = c\tau_1^{-1}$ and let $((t_1, e_1),..., (t_k, e_k))$ be the τ_1-framing of c. Then for any l, $1 \leqslant l \leqslant k$,*

$$\begin{aligned}
&c_1(t_l + 1) \cdots c_1(e_l + 1)\\
&\quad = a^{i+2}0^j, && \text{if there exist } i \text{ and } j \text{ such that } c(t_l) \cdots c(e_l) = a0a^i0^j \in N_1{}^a,\\
&\quad = aa0^j, && \text{if there exist } i \text{ and } j \text{ such that } c(t_l) \cdots c(e_l) = a00^j \in N_2{}^a,\\
&\quad = a0^{j+1}, && \text{if there exist } i \text{ and } j \text{ such that } c(t_l) \cdots c(e_l) = aa0^j \in N_3{}^a,\\
&\quad = a, && \text{if there exist } i \text{ and } j \text{ such that } c(t_l) \cdots c(e_l) = a \in N_4{}^a.
\end{aligned}$$

Furthermore, $c_1(m) = 0$ for any $m < t_1 + 1$ and $e_k + 1 < m$.

Proof. We shall prove that if c_1 is such a configuration that satisfies the equations in the lemma, then $c_1 = c\tau_1^{-1}$ holds. Then the lemma holds by Theorem 2.

Let l be an arbitrary integer such that $1 \leqslant l \leqslant k$. Then, if $c(t_l) = a\ (\neq 0)$, then $c_1(t_l) \neq a$. For, if $c(t_l) = a$ and $1 < l \leqslant k$, then $c(t_{l-1}) \cdots c(e_{l-1}) \in \{a0a^i \mid i \geqslant 1\} \cup \{a0\} \cup \{a\}$ by (3) in Definition 3. From this fact, it is verified that $\sigma_1(c_1(m-1), c_1(m), c_1(m+1)) = c(m)$ for any m, $t_l \leqslant m \leqslant e_l$. Since $c(t_k) \cdots c(e_k) \in \bigcup_{a=1}^{q-1} N_4{}^a$ by (2) in Definition 3, $\sigma_1(c_1(m-1), c_1(m), c_1(m+1)) = c(m)$ for any $m > e_k$. Obviously, $\sigma_1(c_1(m-1), c_1(m), c_1(m+1)) = c(m)$ holds for any $m < t_1$. Q.E.D.

Comment. Given $c \in C_{\mathrm{F}}$, we may define an interval of c as follows. $\eta \in \Sigma^*$ is an interval of c if and only if there exist $i, j\ (i \leqslant j)$ such that

(i) $\eta = c(i)\, c(i+1) \cdots c(j)$, $\eta = a^s 0^t$ for some $s \geqslant 1$, $t \geqslant 0$, $a \in \Sigma - \{0\}$, and

(ii) $c(i-1) \neq c(i)$, and

(iii) $c(j) \neq c(j+1)$ or $t = 1$ and $c(l) = 0$ for any $l > j$.

In view of Lemma 4, we can determine $c\tau_1$ by observing the intervals of c without referring to τ_1. If we consider τ_1 as the transformation of the intervals, it can easily be seen that τ_1 is a natural generalization of the corresponding one used by Yamada and Amoroso in the case with $q = 2$. Note that the interval composed of single a $(a \in \Sigma - \{0\})$ is the only one that appaers in case with $q \geqslant 3$ but does not appear in the case with $q = 2$. A similar comment is also available to the case of τ_2 in view of Lemma 5.

In a way similar to the proof of the above lemma, the next lemma is established.

LEMMA 5. *Let $c_2 = c\tau_2^{-1}$ and let $((e_k, t_k), \ldots, (e_1, t_1))$ be the τ_2-framing of c. Then for any l, $1 \leqslant l \leqslant k$,*

$$
\begin{aligned}
c_2(e_l - 1) \cdots c_2(t_l - 1)
&= 0^j a^{i+2}, && \textit{if there exist } i \textit{ and } j \textit{ such that } c(e_l) \cdots c(t_l) = 0^j a^i 0a \in N_1^{aR},\\
&= 0^j aa, && \textit{if there exist } i \textit{ and } j \textit{ such that } c(e_l) \cdots c(t_l) = 0^j 0a \in N_2^{aR},\\
&= 0^{j+1} a, && \textit{if there exist } i \textit{ and } j \textit{ such that } c(e_l) \cdots c(r_l) = 0^j aa \in N_3^{aR},\\
&= a, && \textit{if there exist } i \textit{ and } j \textit{ such that } c(e_l) \cdots c(t_l) = a \in N_4^{aR}.
\end{aligned}
$$

Furthermore, $c_2(m) = 0$ for any $m < e_k - 1$ and $t_1 - 1 < m$.

The next lemma has been established in the proof of Lemmas 4 and 5.

LEMMA 6. *Let $c_1 = c\tau_1^{-1}$ and let $c_2 = c\tau_2^{-1}$. If t_i is the head of a τ_1-frame of c and $c(t_i) = a$, then $c_1(t_1) \neq a$. Similarly, if t_j is the head of a τ_2-frame of c and $c(t_j) = a$, then $c_2(t_j) \neq a$.*

In what follows, let us put $\pi = \tau_2\tau_1$, Then the next lemma holds.

LEMMA 7. *Let* $c \in C_F$. *If*

$$[c] = \bar{0}\ a_1^2\ a_2^2\ \cdots\ a_s^2\ a_{s+1}\ \text{——}$$

and $c_1 = c\pi^{-1}$, *then* $[c_1]$ *is in one of the forms*

$$\bar{0}\ a_1^2\ a_2^2\ \cdots\ a_s^2\ 0\ \text{——},$$
$$\bar{0}\ a_1^2\ a_2^2\ \cdots\ a_s^2\ a_{s+1}^2\ 0\ \text{——},$$
$$\bar{0}\ a_1^2\ a_2^2\ \cdots\ a_s^2\ a_{s+1}^2\ a_{s+2}\ \text{——},$$

where $s \geqslant 1$, $a_1, \ldots, a_{s+2} \in \{1, 2, \ldots, q-1\}$ *and* $a_i \neq a_{i+1}$ *for any* i, $1 \leqslant i \leqslant s+1$. *Furthermore, if* a_L *in* c *and* c_1 *be in cell* l *and* l_1, *respectively, then* $l = l_1 + 1$.

Proof. From Lemma 4, the relative positions of $[c]$ and $[c\tau_1^{-1}]$ are

$$[c\tau_1^{-1}] = \bar{0}\ 0\ \ a_1\ 0\ \ a_2\ \cdots\ 0\ \ a_s\ 0\ a_{s+1}\ \text{——},$$
$$[c] = \bar{0}\ a_1\ a_1\ a_2\ a_2\ \cdots\ a_s\ a_s\ a_{s+1}\ \text{——}.$$

Let $c' = c\tau_1^{-1}$ and let a_{s+1} in c' be in cell j.

Since $c_1 = c'\tau_2^{-1}$, if j is the head of a τ_2-frame of c, then $c_1(j) \neq a_{s+1}$ from Lemma 6.

Therefore, from Lemma 5, $[c_1]$ is in one of the forms

$$\bar{0}\ a_1^2\ a_2^2\ \cdots\ a_{s+1}^2\ 0\ \text{——},$$
$$\bar{0}\ a_1^2\ a_2^2\ \cdots\ a_{s+1}^2\ a_{s+2}\ \text{——},$$

where $a_{s+1} \neq a_{s+2}$, $a_{s+2} \in \{1, \ldots, q-1\}$. On the other hand, if j is not a head of a τ_2-frame of c', then from Lemma 5 $[c_1]$ is of the form

$$\bar{0}\ a_1^2\ a_2^2\ \cdots\ a_s^2\ 0\ \text{——}.$$

Clearly $l = l_1 + 1$ holds. Q.E.D.

In a similar way to the proof of Lemma 7, we can verify the next lemma.

LEMMA 8. Let $c \in C_F$. If

$$[c] = \text{——}\ a_{s+1}\ 0\ a_s\ 0\ \cdots\ 0\ a_2\ 0\ a_1\ \bar{0}$$

and $c_1 = c\pi^{-1}$, then $[c_1]$ is in one of the forms

$$\text{————}\ 0\ \ \ 0\ a_s\ \cdots\ 0\ a_2\ 0\ a_1\ \bar{0},$$
$$\text{——}\ 0\ \ \ 0\ a_{s+1}\ 0\ a_s\ \cdots\ 0\ a_2\ 0\ a_1\ \bar{0},$$
$$\text{——}\ a_{s+2}\ 0\ a_{s+1}\ 0\ a_s\ \cdots\ 0\ a_2\ 0\ a_1\ \bar{0},$$

where $s \geqslant 1$, $a_1, \ldots, a_{s+2} \in \{1, 2, \ldots, q-1\}$, and $a_i \neq a_{i+1}$ for any i, $1 \leqslant i \leqslant s+1$. Furthermore, if a_R in c and c_1 be in cell r and r_1, respectively, then $r_1 = r + 1$.

Put

$$L^{(q)} = \{a_1^2 a_2^2 \cdots a_i^2 0^m \mid a_1, \ldots, a_i \in \{1, \ldots, q-1\}, i \geqslant 1, \\ m \geqslant 1, a_j \neq a_{j+1} \text{ for any } j, 1 \leqslant j \leqslant i-1\},$$

$$R^{(q)} = \{0^m 0 a_i 0 a_{i-1} \cdots 0 a_1 \mid a_1, \ldots, a_i \in \{1, \ldots, q-1\}, i \geqslant 1, \\ m \geqslant 1, a_j \neq a_{j+1} \text{ for any } j, 1 \leqslant j \leqslant i-1\}.$$

The next lemmas are easily established.

LEMMA 9. *Let* $c \in C_{\mathrm{F}}$. *If*

$$[c] = \bar{0}\eta_1\eta_2 \cdots \eta_r \alpha_1 \alpha_2 \cdots \alpha_i \zeta_s \zeta_{s-1} \cdots \zeta_1 \bar{0},$$
$$\eta_1, \ldots, \eta_r \in L^{(q)}, \zeta_1, \ldots, \zeta_s \in R^{(q)}, \alpha_1, \ldots, \alpha_i \in \Sigma,$$

then

$$[c\pi^{-1}] = \bar{0}\eta_1\eta_2 \cdots \eta_r \beta_1 \beta_2 \cdots \beta_{i+2} \zeta_s \zeta_{s-1} \cdots \zeta_1 \bar{0},$$
$$\beta_1, \ldots, \beta_{i+2} \in \Sigma.$$

Furthermore, if $[c'] = \bar{0}\alpha_1 \cdots \alpha_i \bar{0}$, *then*

$$[c'\pi^{-1}] = \bar{0}\beta_1 \cdots \beta_{i+2}\bar{0}.$$

LEMMA 10. *Let* $c_1 = c\pi^{-1}$ *for* $c \in C_{\mathrm{F}}$. *And let* a_L *in* c *and* c_1 *be in cell* l *and cell* l_1, *respectively. If* $l_1 + 1 = l$, *then* $[c']$ *is in one of the forms*

$$\bar{0}\ a_1 a_1 0 \ \text{——},$$
$$\bar{0}\ a_1 a_1 a_2 \ \text{——},$$

where $a_1, a_2 \in \{1, \ldots, q-1\}$, $a_1 \neq a_2$.

Proof. Let $c' = c\tau_1^{-1}$. *Then* a_L $(=a_1)$ in c' is in cell $l+1$. From Lemma 5 and from $l_1 + 1 = l$, $l + 1$ must be the head of a τ_2-frame of c'. Therefore, $c_1(l+1) \neq a_L$. Thus the lemma is established. Q.E.D.

In a similar way to the proof of the above lemma, the next lemma is established.

LEMMA 11. *Let* $c_1 = c\pi^{-1}$ *for* $c \in C_{\mathrm{F}}$. *And let* a_R *in* c *and* c_1 *be in cell* r *and cell* r_1, *respectively. If* $r_1 = r + 1$, *then* $[c_1]$ *is in one of the forms*

$$\text{——}\ 0\ \ 0 a_1 \bar{0},$$
$$\text{——}\ a_2\ 0 a_1 \bar{0},$$

where $a_1, a_2 \in \{1, \ldots, q-1\}$, $a_1 \neq a_2$.

DEFINITION 5. Let $((t_1, e_1),..., (t_k, e_k))$ be a left framing of c. The *remainder* of c with respect to the left framing $((t_1, e_1),..., (t_k, e_k))$ is c_1 defined as

$$\begin{aligned} c_1(i) &= 0 && \text{if } i \leqslant e_k, \\ &= c(i) && \text{if } i > e_k. \end{aligned}$$

Similarly, the remainder of c with respect to a right framing $((e_k, t_k),..., (e_1, t_1))$ is c_2 defined as

$$\begin{aligned} c_2(i) &= 0 && \text{if } i \geqslant e_k, \\ &= c(i) && \text{if } i < e_k. \end{aligned}$$

In either case, if the framing is the null sequence, we define $c_1 = c$ or $c_2 = c$.

The length of $((t_1, e_1),..., (t_k, e_k))$ is defined to be k. In particular, the length of null sequence is zero.

DEFINITION 6. Let $((t_1, e_1),..., (t_k, e_k))$ be one of the longest left framings of $\in C_F$ with respect to $L^{(q)} \cup \{\lambda\}$, where λ means the null sequence.[5] Let c_1 be the remainder of c with respect to $((t_1, e_1),..., (t_k, e_k))$. Furthermore, let $((e_m', t_m'),..., (e_1', t_1'))$ be one of the longest right framings of c_1 with respect to $R^{(q)} \cup \{\lambda\}$. Then, *π-remainder* of c, denoted by $R(c)$, is defined to be the remainder of c_1 with respect to $((e_m', t_m'),..., (e_1', t_1'))$.

LEMMA 12. *For $c \in C_F$, put $c_i = c\pi^{-i}$, $i = 1, 2,...$. Let a_L and a_R in $R(c_i)$ be l_i and r_i, respectively. Then there exists j such that*

$$l_j = l_{j+1} = \cdots,$$

and

$$r_j = r_{j+1} = \cdots.$$

Proof. The lemma follows from Lemmas 7, 8, 9, 10, and 11. Q.E.D.

If $c(i-1) = c(j+1) \neq a$, $c(i) = c(i+1) = \cdots = c(j) = a$ for $a \in \{1,..., q-1\}$ and $c \in C_F$, then we call $c(i) \cdots c(j)$ a *block* of c. By the *length* of $c \in C_F$, $\lg(c)$, we mean the nonnegative integer $r - l + 1$, where $c(l) = a_L$ and $c(r) = a_R$.

The proof of the next two lemmas is similar to the proof of Lemmas 3 and 4 in our previous paper (Maruoka and Kimura, 1974) and omitted.

LEMMA 13. *Let $c_1 = c\tau_1^{-1}$. Then the following statements hold.*

(i) $\lg(c) = \lg(c_1)$ *holds if and only if the lengths of the rightmost blocks (which we henceforth denote by B_R) in c and c_1 are both odd or both even.*

[5] Because of the ambiguity of e_k, in general the longest left framing may not be determined uniquely. But it is easily seen that c_1 is determined uniquely in this case. This is also the case with $R(c)$ defined later.

(ii) $\lg(c) + 1 = \lg(c_1)$ *holds if and only if the lengths of* B_{R}*'s in* c *and* c_1 *are odd and even, respectively.*

(iii) $\lg(c) - 1 = \lg(c_1)$ *holds if and only if the lengths of* B_{R}*'s in* c *and* c_1 *are even and odd, respectively.*

LEMMA 14. *Let* $c_1 = c\tau_2^{-1}$. *Then the following three statements hold.*

(i) $\lg(c) = \lg(c_1)$ *holds if and only if the lengths of the leftmost blocks (which we henceforth denote by* B_{L}*) in* c *and* c_1 *are both odd or both even.*

(ii) $\lg(c) + 1 = \lg(c_1)$ *holds if and only if the lengths of* B_{L}*'s in* c *and* c_1 *are odd and even, respectively.*

(iii) $\lg(c) - 1 = \lg(c_1)$ *holds if and only if the lengths of* B_{L}*'s in* c *and* c_1 *are even and odd, respectively.*

LEMMA 15. *Let* $c_i = c\tau_1^{-i}$ $(i = 1, 2,...)$ *for* $c \in C_{\mathrm{F}}$. *Then there exists* c_j *such that* $[c] = [c_j]$.

Proof. From Lemma 13, we have $\lg(c_i) \leqslant \lg(c) + 1$ for any c_i. Therefore the lemma holds since τ_1 is a bijection from C_{F} to C_{F}.

In a similar way to the proof of the above lemma, the next lemma is verified.

LEMMA 16. *Let* $c_i = c\tau_1^{-i}$ $(i = 1, 2,...)$ *for* $c \in C_{\mathrm{F}}$. *Then there exists* c_j *such that* $[c] = [c_j]$.

From Lemmas 15 and 16, the next lemma follows.

LEMMA 17. *Let* $c \in C_{\mathrm{F}}$. *For any* $\xi \in \{\tau_1, \tau_2\}^*$, *there exists* $\xi' \in \{\tau_1, \tau_2\}^*$ *such that* $[c\xi] = [c\xi'^{-1}]$.

LEMMA 18. *Let* $c_i = c_0\pi^{-1}$ $(i = 1, 2,...)$ *for* $c_0 \in C_{\mathrm{F}}$. *If* $\lg(R(c_0)) = \lg(R(c_1)) = \cdots$, *then for any* i, $[R(c_i)]$ *is of the form*

$$\bar{0}\ a_1a_1a_1 0 \ \text{——},$$

or

$$\bar{0}\ a_1a_1a_1a_2 \text{——},$$

where $a_1, a_2 \in \{1,..., q - 1\}$, $a_1 \neq a_2$.

Proof. Let a_L and a_R ub $R(c_i)$ be in cell l_i and r_i, respectively. Then from $\lg(R(c_0)) = \lg(R(c_1)) = \cdots$ and from Lemmas 7, 8, 9, 10, and 11, it follows that

$$l_0 = l_1 = \cdots,$$

and

$$r_0 = r_1 = \cdots.$$

Let $c_i' = R(c_i)$, $c_i'' = R(c_i)\tau_1^{-1}$, for $i = 0, 1, \ldots$. Then, from $l_0 = l_1 = l_2$, the relative positions of $[c_0']$, $[c_0'']$, $[c_1']$, $[c_1'']$, and $[c_2']$ are

$$\begin{aligned}
[c_2'] &= \bar{0}\ a_1\ a_1\ \overset{l_0+3}{\overbrace{a_1}}\ \text{——},\\
[c_1''] &= \bar{0}\ 0\ a_1\ 0\ a_1\ \text{——},\\
[c_1'] &= \bar{0}\ a_1\ a_1\ a_1\ \text{——},\\
[c_1''] &= \bar{0}\ 0\ a_1\ \text{———},\\
[c_0'] &= \bar{0}\ a_1\text{————},
\end{aligned}$$

Since $l_0 + 3$ is the head of τ_2-frame of c'', $c_2'(l_0 + 3) \neq a_1$ from Lemma 6. From $l_0 = l_1 = \cdots$, $r_0 = r_1 = \cdots$, and from Lemma 9, for any i there exists c_m' such that $c_i' = c_m'\pi^{-2}$ in a similar way to the proof of Lemma 15. Since the above argument on c_2' and c_0' can also be applied to c_i' and c_m', the lemma is proved. Q.E.D.

LEMMA 19. *Let* $c_i = c_0\pi^{-1}$ $(i = 1, 2, \ldots)$ *for* $c_0 \in C_{\mathrm{F}}$. *If* $\lg(R(c_0)) = \lg(R(c_1)) = \cdots$, *then there exists* $\xi \in \{\tau_1, \tau_2\}^*$ *such that*

$$\lg(R(c_0\xi^{-1})) < \lg(R(c_0)).$$

Proof. In general, $[c_i]$ is of the form

$$[c_i] = \bar{0}\eta_1 \cdots \eta_t\alpha_1 \cdots \alpha_k\zeta_s \cdots \zeta_1\bar{0},$$

where $\eta_1, \ldots, \eta_t \in L^{(q)}$, $\zeta_1, \ldots, \zeta_s \in R^{(q)}$, $\alpha_1, \ldots, \alpha_k \in \Sigma$, $[R(c_i)] = \bar{0}\alpha_1 \cdots \alpha_k\bar{0}$. Let $\eta_t = a_1^2 \cdots a_i^2 0^{m_1}$ and $\zeta_s = 0^{m_2}0a_j' \cdots 0a_1'$. By choosing c_i appropriately we can make m_1 and m_2 sufficiently large. Hence our discussion may be concerned with $R(c_i)$, where c_i is a configuration with sufficiently large m_1 and m_2. From Lemma 18, $[R(c_i)]$ must be in one of the following forms.

Case 1. $[R(c_i)] = 0a_1a_1a_10$ ——, where $a_1 \in \{1, \ldots, q-1\}$.
The relative positions of $[R(c_i)]$, $[R(c_i)\tau_1^{-1}]$, and $[R(c_i)\tau_1^{-1}\tau_2]$ are

$$\begin{aligned}
[R(c_i)] &= \bar{0}\ a_1\ a_1\ a_1\ 0\ \text{———},\\
[R(c_i)\tau_1^{-1}] &= \bar{0}\ \ 0\ a_1\ 0\ \ a_1\ a_1\ \text{——},\\
[R(c_i)\,\tau_1^{-1}\tau_2] &= \bar{0}\ \ 0\ a_1\ a_1\ 0\ \ \text{———}.
\end{aligned}$$

From Lemmas 13 and 14, we have

$$\lg(R(c_i)\,\tau_1^{-1}\tau_2)) < \lg(R(c_i)) = \lg(R(c_0)).$$

Furthermore, since $c_i = c_0\pi^{-i}$, from Lemma 17 there exists $\xi \in \{\tau_1, \tau_2\}^*$ such that $[c_0\xi^{-1}] = [c_i\tau_1^{-1}\tau_2]$. Then, since $R(c_i\tau_1^{-1}\tau_2) = R(R(c_i)\tau_1^{-1}\tau_2)$, we have

$$\begin{aligned} \lg(R(c_0\xi^{-1})) &= \lg(R(c_i\tau_1^{-1}\tau_2)) \\ &= \lg(R(R(c_i)\,\tau_1^{-1}\tau_2)) < \lg(R(c_0)). \end{aligned}$$

Case 2. $[R(c_i)] = 0\, a_1a_1a_1a_2$ ——, where $a_1, a_2 \in \{1,\dots, q-1\}$, $a_1 \neq a_2$.
The relative positions of $c_0' = R(c_i)$, $c_1' = R(c_i)\tau_2$, $c_2' = R(c_i)\tau_2^2$, $c_3' = R(c_i)\tau_2^2\tau_1$, and $c_4' = R(c_i)\tau_2^2\tau_1\tau_2$ are

$$\begin{array}{ll} & \qquad\ \ \underset{\smile}{l} \\ & [c_0'] = \bar{0}\ a_1a_1a_1a_2\ \text{———}, \\ c_1' = c_0'\tau_2, & [c_1'] = \bar{0}\ 0\ a_10\ a_1\ \blacktriangle\text{———}, \\ c_2' = c_1'\tau_2, & [c_2'] = \bar{0}\ 0\ a_1a_1a_1a_1\blacktriangle\text{———}, \\ c_3' = c_2'\tau_1, & [c_3'] = \bar{0}\ a_10\ a_1a_1\ \blacktriangle\text{———}, \\ c_4' = c_3'\tau_2, & [c_4'] = \bar{0}\ a_1a_10\ 0\ a_1\blacktriangle\text{———}. \end{array}$$

Let a_L in c_0' be in cell l. Then $c_1'(l+4) \neq a_1$, $c_2'(l+5) \neq a_1$, $c_3'(l+4) \neq a_1$, $c_4'(l+5) \neq a_1$. These cells whose contents are not a_1 are marked with $\blacktriangle$ in the above patterns. Here we have

$$\begin{aligned} \lg(R(R(c_i)\tau_2^2\tau_1\tau_2) &= \lg(R(c_4')) \\ &< \lg(c_0') = \lg(R(c_i)) = \lg(R(c_0)) \end{aligned}$$

from Lemmas 13 and 14. Since m_1 and m_2 are sufficiently large, we have

$$R(c_i\tau_2^2\tau_1\tau_2) \doteq R(R(c_i)\tau_2^2\tau_1\tau_2).$$

Thus, in a similar way to Case 1, there exists $\xi \in \{\tau_1, \tau_2\}^*$ such that

$$\lg(R(c_0\xi^{-1})) < \lg(R(c_0)). \qquad \text{Q.E.D.}$$

Finally, we have the next lemma.

LEMMA 20 (Theorem 1). *For any $c_0 \in C_F$, there exist $\in\{\tau_1, \tau_2\}^*$ and $c \in C_F$ such that*

$$c = c_0\xi^{-1}$$

and $[c] = \bar{0}\eta_1 \cdots \eta_r\zeta_s \cdots \zeta_1\bar{0}$, *where* $\eta_1,\dots, \eta_r \in L^{(q)}$, $\zeta_1,\dots, \zeta_s \in R^{(q)}$, $r \geqslant 0$, $s \geqslant 0$.

Proof. The theorem follows from Lemmas 7, 8, 9, 10, 11, and 19.

4. An Application of the Decomposition Theorem to the Completeness Problem

In this section we prove that $\mathfrak{A}^{(q,n)}(c_p) = \mathfrak{P}^{(q)}$ holds for any $q \geqslant 2$ and $n \geqslant 3$, using the decomposition theorem (Theorem 1) established in Section 3.

LEMMA 21. *Let*

$$[c] = \bar{0}\eta_1(q-1)\,\eta_2(q-1)\cdots\eta_k(q-1)\,\eta_{k+1}\bar{0},$$

where $\eta_i \in \{0, 1,..., q-2\}^* - \{\lambda\}$ *for* $i, 1 < i < k+1$, *and* $\eta_1,..., \eta_{k+1} \in \{0, 1,..., q-2\}^*$. *Then*

$$[c] \in \mathfrak{A}^{(q,3)}(c_p).$$

Proof. Let

$$\eta_1\eta_2\cdots\eta_{k+1} = \alpha_1\alpha_2\cdots\alpha_j,$$

where $\alpha_1,..., \alpha_j \in \{0, 1,..., q-2\}$. σ_m', $0 \leqslant m \leqslant q-2$, is defined as

$$\begin{aligned} \sigma_m'(\alpha\beta\gamma) &= m, && \text{if } \beta = q-1 \text{ and } \gamma = 0,\\ &= q-1, && \text{if } \alpha\beta\gamma = (q-1)00,\\ &= \beta, && \text{otherwise,} \end{aligned}$$

where $\alpha, \beta, \gamma \in \{0, 1,..., q-1\}$. Let $[c_0] = \bar{0}(q-1)\bar{0}$ and let τ_m' be the global map defined by σ_m'. Then

$$[c_0\tau'_{\alpha_1}\cdots\tau'_{\alpha_j}(\tau_0')^r] = \bar{0}\alpha_1\cdots\alpha_j 0^r(q-1)\bar{0},\ r \geqslant 1.$$

Let $\lg(\eta_i) = n_i$, $1 \leqslant i \leqslant k+1$.[6] From $\mathfrak{A}^{(2,3)}(c_p) = \mathfrak{P}^{(2)}$, it is easily seen that[7]

$$\bar{0}\eta_1\cdots\eta_{k+1}\,0^r(q-1)\,0^{n_2}(q-1)\,0^{n_3}\cdots 0^{n_k}(q-1)\,\bar{0} \in \mathfrak{A}^{(q,3)}(c_p).$$

Let c' be a configuration such that

$$[c'] = \bar{0}\eta_1\cdots\eta_{k+1}\,0^r(q-1)\,0^{n_2}(q-1)\,0^{n_3}\cdots 0^{n_k}(q-1)\bar{0}.$$

[6] By the length of a finite sequence ξ, $\lg(\xi)$, we mean the nonnegative integer k, where $\xi = \beta_1\cdots\beta_k$, $\beta_1,..., \beta_k \in \Sigma$.

[7] Since $\mathfrak{A}^{(2,3)}(c_p) = \mathfrak{P}^{(2)}$, we have $\bar{0}(q-1)\,0^{n_2}(q-1)\,0^{n_3}\cdots 0^{n_k}(q-1)\ \bar{0} \in \mathfrak{A}^{(q,3)}(c_0)$. Furthermore, the global maps used in the derivation of $\bar{0}(q-1)\,0^{n_2}(q-1)\,0^{n_3}\cdots 0^{n_k}(q-1)\bar{0}$ from c_p can be restricted to those whose local maps satisfy the condition that if none of α, β, γ equals to $(q-1)$, then $(\alpha\beta\gamma) = \beta$. Then, since $c_0 \in \mathfrak{A}^{(q,3)}(c_p)$ and $\bar{0}\eta_1\cdots\eta_{k+1}0^r(q-1)\bar{0} \in \mathfrak{A}^{(q,3)}(c_0)$, the desired result can be established.

σ_3 is defined as

$$\begin{aligned}\sigma_3'(\alpha\beta\gamma) &= \alpha, && \text{if } \beta = (q-1) \text{ and } \alpha, \gamma \in \{0, 1,..., q-2\};\\ &= q-1, && \text{if } \gamma = (q-1), \beta \in \{0, 1,..., q-2\} \text{ and } \alpha = \{0, 1,..., q-1\};\\ &= \beta, && \text{otherwise.}\end{aligned}$$

Then, since $n_2 \geqslant 1,..., n_k \geqslant 1$,

$$[c] = [c' \tau_3^{r+n_2+\cdots+n_{k+1}}],$$

where τ_3 is the global map defined by σ_3. Thus, $[c] \in \mathfrak{A}^{(q,3)}(c_p)$. Q.E.D.

LEMMA 22. *Let* $[c]$ *be of the form*

$$[c] = \bar{0}\eta_1 \cdots \eta_r \zeta_s \cdots \zeta_1 \bar{0},$$

where $\eta_1,..., \eta_r \in L^{(q)}$, $\zeta_1,..., \zeta_s \in R^{(q)}$, $r \geqslant 0$, $s \geqslant 0$, $q \geqslant 2$. *Then*, $[c] \in \mathfrak{A}^{(q,3)}(c_p)$.

Proof. The lemma holds for $q = 2$ from $\mathfrak{A}^{(2,3)}(c_p) = \mathfrak{P}^{(2)}$ (Maruoka and Kimura). So assume $q \geqslant 3$. Let us define c' from c as

$$\begin{aligned}c'(i) &= 1, && \text{if } c(i) = c(i+1) = q-1,\\ &= c(i), && \text{otherwise.}\end{aligned}$$

Then, from Lemma 21 we have $[c'] \in \mathfrak{A}^{(q,3)}(c_p)$. Let σ_4 be defined as

$$\begin{aligned}\sigma_4(\alpha\beta\gamma) &= q-1, && \text{if } \gamma = q-1 \text{ and } \beta = 1,\\ &= \beta, && \text{otherwise.}\end{aligned}$$

Clearly, $[c'\tau_4] = [c]$, where τ_4 is specified by σ_4. Thus, we have $[c] \in \mathfrak{A}^{(q,3)}(c_p)$. Q.E.D.

THEOREM[8] 23. *For any integer* $q \geqslant 3$, $\mathfrak{A}^{(q,3)}(c_p) = \mathfrak{P}^{(q)}$.

Proof. The theorem follows from Lemmas 20 and 22. Q.E.D.

COROLLARY 24. *For any* $q \geqslant 2$ *and* $n \geqslant 3$,

$$\mathfrak{A}^{(q,n)}(c_p) = \mathfrak{P}^{(q)}.$$

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[8] This theorem has been obtained by Nasu and Honda independently employing graph techniques.

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